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THE PROJECTION CONSTANT OF FINITE-DIMENSIONAL SPACES WHOSE UNCONDITIONAL BASIS CONSTANT IS 1

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ABSTRACT

The relations of the projection constant $\lambda(E)$ and the isomorphic distance $d(E, l_n^\infty)$ of finite-dimensional spaces E whose unconditional basis constant is 1 are investigated. It turns out that both are proportional to the norm of a certain vector in E.

1. Preliminaries

In this paper we shall use standard notations of the theory of Banach spaces. Let us just recall that the isomorphic distance between Banach spaces E and F is defined by

$$d(E, F) := \inf_J (\|J\| \|J^{-1}\|),$$

where the inf is taken over all isomorphism J of E onto F. If there are no isomorphisms we define $d(E, F) := \infty$. Every Banach space E can be embedded isometrically in a space l_d^∞ for a suitable d. Let namely $d \subset E'$ be a subset so that the closed convex hull of d is the unit sphere of E' and define $I: E \to l_d^\infty$ by

$$I(x) = (\langle z, x \rangle)_{z \in d}. \tag{1}$$

Let E_0 be the image of an embedding map in a space l_d^∞. Then we define the projection constant of a Banach space E by

$$\lambda(E) := \inf\{\|P\| \mid P: l_d^\infty \to E_0,\ P \text{ projection}\}.$$

We define $\lambda(E) := \infty$ if there is no projection. $\lambda(E)$ is well-defined because it is independent of how E is embedded. In the following we only deal with finite-dimensional spaces E. We define [3] the coordinate asymmetry of a basis $(e_i)_{i=1,\cdots,n}$ of E by

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$$\chi((e_i)_{i=1,\cdots,n}) := \sup_{\substack{(a_i)_{i=1,\cdots,n} \\ \varepsilon_i = \pm 1}} \frac{\left\| \sum_{i=1}^n \varepsilon_i a_i e_i \right\|}{\left\| \sum_{i=1}^n a_i e_i \right\|}$$

The coordinate asymmetry or the unconditional basis constant of E is

$$\chi(E) = \inf_{(e_i)_{i=1,\cdots,n} \subset E} \chi((e_i)_{i=1,\cdots,n}).$$

Lindenstrauss and Pełczyński [5] proved that

$$d(E, l_n^\infty) \leqq K_G^2 \lambda(E)^2 \chi(E)^2,$$

where K_G is the Grothendieck constant. We are dealing in this paper with spaces with $\chi(E) = 1$. As the main result we get (Corollary 2): Suppose $(e_i)_{i=1,\cdots,n}$ is a normalized basis of Banach space E with $\chi((e_i)_{i=1,\cdots,n}) = 1$ and $\|\sum_{i=1}^n a_i e_i\| \leqq (\sum_{i=1}^n a_i^2)^{\frac{1}{2}}$. Then $\lambda(E)$ and $d(E, l_n^\infty)$ are, up to a universal constant, equal to $\|\sum_{i=1}^n e_i\|$.

By renorming a basis whose coordinate asymmetry is 1 we can restrict ourselves to the case that E is $\mathbf{R}^n$ with a norm $\| \ \|_E$, $\| \ \|_\infty \leqq \| \ \|_E \leqq \| \ \|_1$, and the unit vectors are a basis with coordinate asymmetry equal to 1. The most important analytical tool we use is the Khintchin-inequality for $p = 1$ [7]

$$(2) \qquad C_1 \|(a_i)_{i=1,\cdots,n}\|_2 \leqq 2^{-n} \sum_{j=1}^{2^n} \left| \sum_{i=1}^n r_i(j) a_i \right| \leqq \|(a_i)_{i=1,\cdots,n}\|_2$$

satisfied by the Rademacher-functions

$$r_i : \{1, \cdots, 2^n\} \to \{+1, -1\}, \qquad i = 1, \cdots, n.$$

Szarek [7] proved that $C_1 = 1/\sqrt{2}$.

2. An estimation of the projection constant

The aim of this paragraph is to prove the following theorem.

THEOREM 1. *Let E be $\mathbf{R}^n$ with the norm $\| \ \|_E$, $\| \ \|_\infty \leqq \| \ \|_E \leqq \| \ \|_1$, and let the unit vectors be a basis whose coordinate asymmetry is 1. Then*

$$(3) \qquad \begin{aligned} \frac{1}{\sqrt{2}} \|(1, \cdots, 1)\|_E \left(\min_{\|y\|_E = 1} \|y\|_2 \right) &\leqq \lambda(E) \\ &\leqq \min\{\sqrt{n}, \|(1, \cdots, 1)\|_E\} \end{aligned}$$

and

$$\frac{1}{\sqrt{2}}\left(\max_{\|z\|_{E'}=1}\|z\|_1\right)\left(\max_{\|z\|_{E'}=1}\|z\|_2\right)^{-1}\leqq \lambda(E)$$

(4)

$$\leqq \min\left\{\sqrt{n},\ \max_{\|z\|_{E'}=1}\|z\|_1\right\}.$$

Before we prove the theorem we just make some remarks on the proof and we give some corollaries.

In the proof of Theorem 1 we can restrict ourselves to spaces with polyhedral unit spheres by using a standard approximation argument. Therefore we can choose a special embedding map (1). Moreover we can find a finite set M of functionals with positive components so that the convex hull of

$$\tilde{M}=\{(z(i)r_i(j))_{i=1,\cdots,n}\mid z\in M,\ j=1,\cdots,2^n\},$$

where r_i, $i=1,\cdots,n$ are the first n Rademacher-functions, is the unit ball of the dual space. As a special embedding map we choose I defined by

(5) $$I(y):=(((\langle(z(i)r_i(j))_{i=1,\cdots,n},y\rangle)_{j=1,\cdots,2^n})_{z\in M}.$$

We define

(6) $$x_i:=I(e_i)=((z(i)r_i(j))_{j=1,\cdots,2^n})_{z\in M}.$$

COROLLARY 2. *Let E be $\mathbf{R}^n$ with the norm $\|\ \|_E$ and $\|\ \|_\infty\leqq\|\ \|_E\leqq\|\ \|_2$. Suppose that the unit vectors are a basis whose coordinate asymmetry is* 1. *Then*

(7) $$\frac{1}{\sqrt{2}}\|(1,\cdots,1)\|_E\leqq\lambda(E)\leqq \mathrm{d}(E,l_n^\infty)\leqq\|(1,\cdots,1)\|_E.$$

PROOF. Trivial.

COROLLARY 3. *Let E be $\mathbf{R}^n$ with the norm $\|\ \|_E$ and $\|\ \|_\infty\leqq\|\ \|_E\leqq\|\ \|_1$. Suppose that the unit vectors are a basis whose coordinate asymmetry is* 1. *Then*

$$\frac{1}{\sqrt{2}}(\|(1,\cdots,1)\|_E)^{\frac12}\leqq\lambda(E)\leqq \mathrm{d}(E,l_n^\infty)\leqq\|(1,\cdots,1)\|_E.$$

PROOF. In order to verify the left-hand side inequality we just have to observe

$$\min_{\|y\|_E=1}\|y\|_2=\left(\max_{\|y\|_{E'}=1}\|y\|_2\right)^{-1}$$

$$=\left(\max_{\|y\|_{E'}=1}\|y\|_2^2\right)^{-\frac12}.$$

Because of $\| \|_\infty \leqq \| \|_E$ we get

$$\geqq \left(\max_{\|y\|_{E'}=1} \|y\|_1 \right)^{-\frac{1}{2}}$$

$$= (\|(1, \cdots, 1)\|_E)^{-\frac{1}{2}}.$$

PROOF OF THEOREM 1. (3) is just the dual formulation of (4). The right side estimation in (4) is obtained by considering the identity map $I : l_n^\infty \to E$ and getting

$$\|I\| \|I^{-1}\| = \max_{\|z\|_{E'}=1} \|z\|_1$$

and a result of Kadec and Snobar [4].

We prove now the left side inequality. Because of an approximation argument we can restrict ourselves to spaces E with a polyhedral unit sphere. We use the embedding map (5) and get as a representation for projections

$$P(x) = \sum_{i=1}^{n} < f_i, |x > x_i \tag{8}$$

where $(f_i)_{i=1,\cdots,n}$ are functionals that are biorthogonal with respect to the $(x_i)_{i=1,\cdots,n}$ defined by (6). The components of a functional f_i, $i = 1, \cdots, n$, are denoted by f_i^z, $z \in \tilde{M}$, so that we have

$$\langle x_j, f_i \rangle = \sum_{z \in \tilde{M}} z(j) f_i^z = \delta_{ji}.$$

Then

$$\|P\| = \max_{\|x\|_\infty=1} \left\| \sum_{i=1}^{n} \langle f_i, x \rangle x_i \right\|_\infty$$

$$= \max_{\|x\|_\infty=1} \max_{z \in \tilde{M}} \left| \sum_{i=1}^{n} \langle f_i, x \rangle z(i) \right|$$

$$= \max_{z \in \tilde{M}} \max_{\|x\|_\infty=1} \left| \left\langle \sum_{i=1}^{n} z(i) f_i, x \right\rangle \right|$$

$$= \max_{z \in \tilde{M}} \sum_{w \in \tilde{M}} \left| \sum_{i=1}^{n} z(i) f_i^w \right|.$$

From the Khintchin-inequality (2) we get that

$$\|P\| \geqq C_1 \max_{z \in M} \sum_{w \in \tilde{M}} \|(z(i) f_i^w)_{i=1,\cdots,n}\|_2. \tag{9}$$

On the other hand $(f_i)_{i=1,\cdots,n}$ are orthogonal to $(x_i)_{i=1,\cdots,n}$. Thus

$$\sum_{w\in\tilde{M}} f_i^w w(i) = 1 \quad \text{for all} \quad i = 1,\cdots,n.$$

Hence

$$\sum_{w\in\tilde{M}} f_i^w w(i) z(i) = z(i) \quad \text{for all} \quad i = 1,\cdots,n \quad \text{and} \quad z \in M$$

and

$$\begin{aligned}\|z\|_1 &\leqq \sum_{w\in\tilde{M}} |\langle (f_i^w z(i))_{i=1,\cdots,n}, w\rangle| \\ &\leqq \sum_{w\in\tilde{M}} \|w\|_2 \|(f_i^w z(i))_{i=1,\cdots,n}\|_2 \\ &\leqq \left(\max_{w\in\tilde{M}} \|w\|_2\right) \sum_{w\in\tilde{M}} \|(f_i^w z(i))_{i=1,\cdots,n}\|_2.\end{aligned}$$

Therefore

$$\|z\|_1 \left(\max_{\|w\|_{E'}=1} \|w\|_2\right)^{-1} \leqq \sum_{w\in\tilde{M}} \|(f_i^w z(i))_{i=1,\cdots,n}\|_2.$$

With (9) we get

$$\|P\| \geqq C_1 \max_{\|z\|_{E'}=1} \|z\|_1 \left(\max_{\|z\|_{E'}=1} \|z\|_2\right)^{-1}.$$

With that Theorem 1 is proved.

REMARK 4. In Theorem 1 we can use a weaker hypothesis than $\chi(E) = 1$. We just need that there is a $z \in E'$ with $\|z\|_1 = \max_{\|y\|_{E'}=1} \|y\|_1$ and

$$\|(z(i) r_i(j))_{i=1,\cdots,n}\|_{E'} = 1 \quad \text{for all} \quad j = 1,\cdots,2^n.$$

Now we want to apply our results to the n-dimensional Orlicz spaces l_n^H. Let H be a convex function

$$H: \mathbf{R} \to \mathbf{R}^+$$

with $H(0) = 0$, $H(1) = 1$, $H(x) = H(|x|)$ for all $x \in \mathbf{R}$, and $H(x) \neq 0$ for all $x \neq 0$. l_n^H is $\mathbf{R}^n$ with the norm generated by the unit ball.

$$\left\{ y \in \mathbf{R}^n \,\middle|\, \sum_{i=1}^n H(y(i)) \leqq 1 \right\}.$$

COROLLARY 5. (i) *Let* $H(x) \leqq x^2$. *Then*

$$\frac{1}{\sqrt{2}}\left(H^{-1}\left(\frac{1}{n}\right)\right)^{-1} \leqq \lambda(l_n^H) \leqq d(l_n^H, l_n^\infty) \leqq \left(H^{-1}\left(\frac{1}{n}\right)\right)^{-1}.$$

(ii) *Let H be differentiable and* $H'(x)/x$ *be strictly monotone on* $(0,1)$. *Moreover, let* $(\sqrt{n}H^{-1}(1/n))_{n\in\mathbb{N}}$ *be a decreasing sequence. Then*

$$\sqrt{\frac{n}{2}} \leqq \lambda(l_n^H) \leqq \sqrt{n}.$$

(i) is an immediate consequence of Corollary 2 and (ii) can be verified by applying Theorem 1 and using Lagrangian multipliers. Let us note that Corollary 5 covers the cases of l_n^p, $1 \leqq p \leqq \infty$, which are already treated by Gordon [1], Grünbaum [2], and Rutovitz [6]. (ii) improves the inequality achieved by Gordon [1] slightly.

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